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ON THE STABILITY OF FLOW OF PERFECT GAS IN A CHANNEL WITH A CLOSING COMPRESSION SHOCK AND SIMULTANEOUS REFLECTION OF ACOUSTIC AND ENTROPY WAVES FROM THE OUTLET CROSS SECTION

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The stability of flow of an inviscid non-heat-conducting gas in a channel with closing compression shock is investigated in the case when the boundary condition at the channel outlet is specified in the form of a linear relationship between the unsteady perturbation of the left-hand Riemannian invariant which defines the reflected acoustic wave and that of the right-hand Riemannian invariant and the entropy function which define waves that reach the outlet from the channel side. The investigation results in the determination of stability region in the plane of reflection coefficients. The analysis is based on the " D-separation" method widely used in the theory of automatic control [1, 2] and on the stability conditions obtained in [3] for the case when one of the reflection coefficients is zero. The investigation is carried out as in [3] in a "quasi-cylindrical" approximation.

1. The quasi-cylindrical approximation used here was first proposed by G. G. Chernyi in 1953. However the final formulas of this approximation presented in [3] do not provide a sufficiently clear concept of conditions that limit its validity. Because of this, we shall consider this question in greater detail.

Let us consider the flow of a perfect gas in a channel whose cross section area F is a known function of the x-coordinate measured along the channel axis. In what follows we investigate the stability of a steady flow with a closing compression shock in which the stream velocity varies from supersonic (to the left of the shock) to subsonic. As the reference point for x we take the stationary position of the shock and as the characteristic length we take distance of that section from the channel outlet. Hence the outlet section is at $x = 1$. The reduction of remaining parameters to the dimensionless form is carried out as in [3], except that the critical velocity and density of the steady stream to the left of the shock are taken as the corresponding characteristic parameters.

Equations of one-dimensional unsteady flow are linearized in the usual manner, and pressure p, density ρ and velocity u are represented as

$$u(x, t) = U(x)[1 + u_n(x, t)]$$

where t is the time, capital letter denotes respective steady quantity and the subscript n appears at the relative unsteady perturbation of the particular parameter.

After linearization of equations defining the flow of a perfect gas with adiabatic exponent $\varkappa$, we obtain the following system:

$$D^+R / Dt = a_{11}R + a_{12}L + a_{13}S \tag{1.1}$$
$$D^-L / Dt = a_{21}R + a_{22}L + a_{23}S, \quad DS / Dt = 0$$

$$R = \frac{1}{2}\left(u_n + \frac{p_n}{\varkappa M}\right), \quad L = \frac{1}{2}\left(u_n - \frac{p_n}{\varkappa M}\right)$$
$$S = p_n - \varkappa \rho_n, \quad \frac{D^+}{Dt} = \frac{\partial}{\partial t} + (U + A)\frac{\partial}{\partial x}$$
$$\frac{D^-}{Dt} = \frac{\partial}{\partial t} + (U - A)\frac{\partial}{\partial x}, \quad \frac{D}{Dt} = \frac{\partial}{\partial t} + U\frac{\partial}{\partial x}$$

where R, L and S are the unsteady perturbations of the right- and left-hand Riemann invariants, and of the entropy function, respectively; D^+ / Dt, D^- / Dt and D / Dt are operators of differentiation along the characteristics of the first and second sets, and of the particle trajectory; A is the steady-state speed of sound, and $M = U / A$ is the Mach number. The coefficients a_{ij} are determined by formulas

$$a_{11} = A\left[\frac{(\varkappa - 1) M - 2}{(\varkappa - 1) M^2 + 2} - \frac{1}{2}\left(1 + \frac{1}{M}\right)\right] M' \tag{1.2}$$
$$a_{12} = A\left[\frac{1}{2}\left(1 + \frac{1}{M}\right) - \frac{(\varkappa - 1) M + 2}{(\varkappa - 1) M^2 + 2}\right] M'$$
$$a_{13} = a_{23} = \frac{A}{2\varkappa}\left(1 + \frac{\varkappa - 1}{2} M^2\right)^{-1} M'$$
$$a_{21} = a_{11} + AM', \quad a_{22} = a_{12} - AM'$$
$$M' = M\,[2 + (\varkappa - 1)M^2]\,(\ln F)' / 2\,(M^2 - 1)$$

where primes denote total derivatives with respect to x.

It will be seen from (1.1) that the trajectories of acoustic waves propagating respectively down- and upstream (the R- and L-waves) and entropy waves (S-waves) are defined by the equations

$$dx / dt = U + A, \quad dx / dt = U - A, \quad dx / dt = U$$

The time taken by these waves to move from the shock to the channel outlet (or in the reverse direction) are, respectively,

$$\tau_R = \int_0^1 \frac{dx}{U + A}, \quad \tau_L = \int_0^1 \frac{dx}{A - U}, \quad \tau_S = \int_0^1 \frac{dx}{U}$$

It is not difficult to show that in the considered case $(0 < U < A)$ the inequality $\tau_L > \tau_R$ is valid. If the stream ahead of the shock (and consequently, also behind it) is close to sonic $(A \approx U \approx 1)$ and $|M'| \ll 1$ throughout section $0 \leqslant x \leqslant 1$, then $\tau_L \gg 1$, and τ_S and τ_R are quantities of the order of unity. As implied by formulas (1.2) all coefficients a_{ij} are proportional to M'. Hence, if the maximum of $|M'|$ is such that the product $|M'|\,\tau_L \ll 1$, then, as seen from (1.1), the relative increments of invariants R and L for the motion of R- and L-waves along the channel are, also considerably smaller than unity. Moreover, in that case the relative vibrations of the quantities U, $U + A$ and $U - A$ are also small. The latter makes it possible to substitute for these quantities their values for $x \to +0$, i.e. to the left of the closing shock (the quantities at $x \to +0$ will be denoted by subscript "plus"). As the result we obtain the following quasi-cylindrical approximation equations:

$$R\,(x, t) = R\,(\xi), \quad L\,(x, t) = L\,(\eta), \quad S\,(x, t) = S\,(\zeta) \tag{1.3}$$
$$\xi = x - (U_+ + A_+)t, \quad \eta = x - (U_+ - A_+)t, \quad \zeta = x - U_+ t$$

$$\tau_R = 1 / (U_+ + A_+), \quad \tau_L = 1 / (A_+ - U_+), \quad \tau_S = 1 / U_+$$

where ξ, η and ζ are characteristic variables which remain constant along related characteristics. The condition of validity of formulas (1.3) is expressed in the form of inequality

$$\tau_L \max_{\substack{0 \leqslant x \leqslant 1 \\ j=1,2,3}} |a_{2j}| \ll 1 \tag{1.4}$$

We remind that coefficients a_{ij} are proportional to M' and that the coefficients of proportionality are of the order of unity. Note that when M tends to unity, then, in accordance with (1.3), (1.4) and the last of formulas (1.2), the validity of the quasi-cylindrical approximation requires that the derivative $(\ln F)'$ must decrease as $(1 - M)^2$.

Equations (1.3) make it possible to link the parameters of the perturbed stream at the shock stationary position ($x = 0$) to those at the channel outlet ($x = 1$) where boundary conditions are specified. Boundary conditions for $x = 0$ are obtained by linearizing the relationships at the shock. They are of the form

$$R_+ = \varphi L_+ - \psi Y x_s, \quad S_+ = \varphi' L_+ - \psi' Y x_s \tag{1.5}$$

$$x_s^{\boldsymbol{\cdot}} = \mu L_+ - \beta Y x_s$$

$$\varphi = \frac{(1 - 2M_+) M_-^2 + 1}{(1 + 2M_+) M_-^2 + 1}, \quad \psi = \frac{[(K - 1) N + E] M_+}{(1 + 2M_+) N + M_+^2}$$

$$\varphi' = M_+ (1 - \varphi)(1 - N) \varkappa / N, \quad \psi' = [E - M_+ (1 - N) \psi] \varkappa / N$$

$$\mu = \frac{1 - M_+ - (1 + M_+) \varphi}{(U_- - U_+) M_+}, \quad \beta = U_- - \frac{(1 + M_+) \psi}{(U_- - U_+) M_+}$$

$$K = \frac{U_-}{U_+}, \quad N = M_+^2 M_-^2, \quad E = \frac{\varkappa - 1}{\varkappa + 1} (M_+^2 - M_-^2)$$

$$Y = (d \ln F / dx)_{x=0}$$

where $x = x_s(t)$ is the shock trajectory, a dot denotes differentiation with respect to t, and the subscript "minus" denotes parameters ahead of the shock. In deriving (1.5) we assumed, as in [3], that no perturbations reach the closing shock from the upstream side. Certain differences in formulas for coefficients (1.5) from the expressions in [3] are due to the somewhat different notation.

At the channel outlet the velocity is subsonic. Hence the R- and S-waves arriving there can be reflected in the form of L-waves. In accordance with this we specify the condition at $x = 1$ as

$$L = \chi R + \chi' S \tag{1.6}$$

where χ and χ' are reflection coefficients which are assumed to be specified.

2. The statement of the problem of flow stability in a channel, which is defined by Eqs. (1.3) with boundary conditions (1.5) and (1.6), is not different from that in [3]. In accordance with (1.3) and (1.5) the complete picture of the development of flow is provided by the analysis of two functions of time: $x_s(t)$ and $L_+(t)$ which are determined by the system of two differential-difference equations

$$x_s^{\boldsymbol{\cdot}}(t) = \mu L_+(t) - \beta Y x_s(t) \tag{2.1}$$

$$L_+(t) = \varphi_0 L_+(t - \tau) + \varphi_0' L_+(t - \tau') -$$

$$\psi_0 Y x_s(t-\tau) - \psi_0' Y x_s(t-\tau')$$
$$\varphi_0 = \chi\varphi, \quad \varphi_0' = \chi'\varphi', \quad \psi_0 = \chi\psi, \quad \psi_0' = \chi'\psi'$$
$$\tau = \tau_L + \tau_R, \quad \tau' = \tau_L + \tau_S$$

The first of Eqs. (2.1) is the same as the third of (1.5) with indication of arguments, and the second is obtained from (1.3), (1.5) and (1.6) with allowance for the reflection of various waves from the shock and from the channel outlet (compare with [3]).

The behavior of solution of system (2.1) when $t \to \infty$, is determined by the disposition in the complex plane λ of roots of the characteristic equation of that system which is of the form [4]

$$H(\lambda) \equiv (\lambda + \beta Y)(1 - \varphi_0 e^{-\tau\lambda} - \varphi_0' e^{-\tau'\lambda}) + \mu Y(\psi_0 e^{-\tau\lambda} + \psi_0' e^{-\tau'\lambda}) = 0 \qquad (2.2)$$

For all solutions of (2.1) to be bounded when $t \to \infty$ (i.e. for the reference steady flow to be stable) it is sufficient that the following conditions are satisfied: first, the real parts of all roots of (2.2) must not be positive and, second, each root of (2.2) with zero real parts must be simple. The first of these is also the necessary condition. The disposition of roots of the characteristic equation depends on its coefficients which are uniquely determined by $\varkappa$, Y, U_-, and on the reflection coefficients χ and χ'. For a specific gas (i.e. a gas with a fixed adiabatic exponent $\varkappa$) and a specified shape of the channel (and, consequently, also Y) the analysis of stability reduces to the determination of a surface which in the space $U_-\chi\chi'$ binds the stability region. The intersection of that surface with any arbitrary plane $U_- = \text{const}$ with $1 < U_- < U_{\max} = \sqrt{(\varkappa+1)/(\varkappa-1)}$, represents the stability region boundary in the $\chi\chi'$-plane.

For any arbitrary fixed $\varkappa$, Y and U_- that boundary is constructed as follows.

Since the number n of roots of Eq. (2.2) with positive real parts depends on χ and χ', hence different n correspond to different regions of the $\chi\chi'$-plane. If $D(n)$ represents the related region of the $\chi\chi'$-plane, then $D(0)$ coincides with the stability region. The used here "D-separation" method comprises the construction of lines ("Nyquist curves") which in the $\chi\chi'$-plane separate regions $D(n)$ and $D(n+1)$.

In the case of continuous dependence of roots λ on parameters χ and χ' the equation of the indicated curves is defined by $\operatorname{Re}\lambda = 0$, and their construction is carried out as follows [1, 2]. We substitute $\lambda = i\omega$, where ω is a real number, into (2.2), and define $H(i\omega)$ by $H(i\omega) = F(\omega) + iG(\omega)$. We equate functions $F(\omega)$ and $G(\omega)$ to zero and determine from the obtained equations, which are linear with respect to χ and χ', the coefficients in terms of functions of ω, and obtain the parametric representation of the sought curves as $\chi = \chi(\omega)$ and $\chi' = \chi'(\omega)$, where the functions in the right-hand sides are known and ω varies from 0 to ∞. If $\Delta(\omega)$ is the determinant of coefficients at χ and χ', and $A(\omega)$ and $A'(\omega)$ are determinants obtained from $\Delta(\omega)$ by the substitution of the column of free terms for one of its columns, then

$$\chi(\omega) = A(\omega)/\Delta(\omega), \quad \chi'(\omega) = A'(\omega)/\Delta(\omega) \qquad (2.3)$$
$$\Delta(\omega) = (Y^2 QQ' + \omega^2\varphi\varphi')\sin\alpha\omega + Y\omega(Q\varphi' - Q'\varphi)\cos\alpha\omega$$
$$A(\omega) = (\omega^2\varphi' - Y^2\beta Q')\sin\tau'\omega - Y\omega(Q' + \beta\varphi')\cos\tau'\omega$$
$$A'(\omega) = (Y^2\beta Q - \omega^2\varphi)\sin\tau\omega + Y\omega(Q + \beta\varphi)\cos\tau\omega$$
$$Q = \mu\psi - \beta\varphi, \quad Q' = \mu\psi' - \beta\varphi', \quad \alpha = \tau' - \tau$$

For $\omega = 0$ the determinants Δ, A and A' vanish and instead of (2.3) we obtain a "singular straight line" [1, 2]

$$\chi Q + \chi' Q' + \beta = 0 \tag{2.4}$$

which is also the Nyquist curve.

It is not possible to to construct Nyquist curves throughout the whole range of " frequencies" $0 < \omega < \infty$. It is, therefore, necessary to determine the stability region of the "simplified" characteristic equation

$$\chi\varphi e^{-\tau\lambda} + \chi'\varphi' e^{-\tau'\lambda} = 1 \tag{2.5}$$

which is obtained from (2.2) for $|\lambda| \gg 1$.

We shall show that the rhombus

$$|\chi\varphi \pm \chi'\varphi'| < 1 \tag{2.6}$$

is to be taken as the stability region of Eq. (2.5).

First of all, it follows from (2.5) that along the diagonals of the rhombus (2.6) $\mathrm{Re}\lambda < 0$, i.e. that $\chi' = 0$ when $|\chi\varphi| < 1$, and $\chi = 0$ when $|\chi'\varphi'| < 1$, while along the remaining parts of axes χ and χ', $\mathrm{Re}\,\lambda > 0$. It can be further shown that the Nyquist curves of Eq. (2.5) do not enter the above rhombus for any τ and τ' The above clearly implies that the stability region of Eq. (2.5) is never smaller than the rhombus (2.6). It is, however, possible to give examples showing that for rational τ'/τ the stability region in related cases is greater than the rhombus (2.6). For example, when $\tau'/\tau = 1$, the whole band $|\chi\varphi + \chi'\varphi'| < 1$, which contains the considered rhombus represents such region. However for irrational τ'/τ the stability region coincides with rhombus (2.6). We shall prove this using, as in [5], Kroneker's theorem [6].

Let τ'/τ be irrational. We rewrite (2.5) in the form

$$\begin{gathered}\zeta_1 = (1 - b\zeta_2)/a, \quad a = |\chi\varphi| e^{-\tau\sigma}, \quad b = |\chi'\varphi'| e^{-\tau'\sigma}, \quad \zeta_1 = e^{-i\theta_1} \\ \zeta_2 = e^{-i\theta_2}, \quad \theta_1 = \tau\omega - \arg(\chi\varphi) + 2\pi n, \quad \theta_2 = \tau'\omega - \arg(\chi'\varphi') + \\ 2\pi m \quad (\sigma = \mathrm{Re}\,\lambda, \ \omega = \mathrm{Im}\,\lambda)\end{gathered} \tag{2.7}$$

where n and m are integers or zero.

We have to determine the roots of Eq. (2.7), i.e. the pairs of σ and ω which satisfy that equation or two equivalent to it real relationships

$$\begin{gathered}1 = R(\sigma, \nu) \equiv |1 - 2b\nu + b^2| / a \\ \theta_1 = \arg[(1 - b\zeta_2)/a] \quad (\nu = \cos\theta_2)\end{gathered} \tag{2.8}$$

We take any arbitrary ν from the interval $-1 \leqslant \nu \leqslant 1$ and determine $\theta_{2\nu}$ and $\zeta_{2\nu}$ for that value of ν, then from (2.8) we obtain σ_ν and $\theta_{1\nu}$. All relationships which follow from (2.5), with the exception of the last two equalities in (2.7), are then satisfied. These equalities can be satisfied in the case of irrational τ'/τ with any required accuracy by a suitable selection of ω, n and m. This statement from the Kroneker's theorem by which the inequalities

$$\begin{gathered}|\tau\omega - \theta_1^\circ + 2\pi n| < \delta, \qquad |\tau'\omega - \theta_2^\circ + 2\pi m| < \delta \\ \theta_1^\circ = \theta_{1\nu} + \arg(\chi\varphi), \qquad \theta_2^\circ = \theta_{2\nu} + \arg(\chi'\varphi')\end{gathered} \tag{2.9}$$

have common real solutions for irrational τ'/τ and any arbitrarily small δ. Note that for rational τ'/τ it is not generally possible to satisfy inequalities (2.9) for any arbitrary ν.

Let us now take some ω_ν which satisfies the indicated inequalities and introduce $\varepsilon_{i\nu}$ such that

$$\varepsilon_{1\nu} = \tau\omega_\nu - \theta_1^\circ + 2\pi n, \qquad \varepsilon_{2\nu} = \tau'\omega_\nu - \theta_2^\circ + 2\pi m \tag{2.10}$$

where n and m are integers which with the selected ω ensure the satisfaction of inequalities (2.9). Hence $|\varepsilon_{i\nu}| < \delta$. The set of roots σ_ν and ω_ν owing to the arbitrariness of the selection of ν in (2.8) can only be greater than the set of roots (2.7).

We shall prove that σ_ν and ω_ν determined by the described procedure are as required close to the real and imaginary parts of some root of (2.5) or (2.7) and that, consequently, the inequality $\sigma_\nu \leqslant 0$ for $-1 \leqslant \nu \leqslant 1$ is equivalent to the condition of nonpositivity of real parts of all roots of the characteristic equation (2.5) or (2.7). To prove this we represent the roots of (2.5) in the form

$$\lambda = \sigma_\nu + \Delta\sigma + i(\omega_\nu + \Delta\omega) \tag{2.11}$$

Substituting (2.11) into (2.5) and expanding in $\Delta\sigma$ and $\Delta\omega$ on the assumption of their smallness, and taking into consideration that σ_ν and ω_ν satisfy inequalities (2.8) and (2.10), we obtain two linear equations for determining $\Delta\sigma$ and $\Delta\omega$. In the right-hand sides of these equations we have linear combinations of $\varepsilon_{i\nu}$. It follows from this that $|\Delta\sigma| < k_1\delta$ and $|\Delta\omega| < k_2\delta$, where $k_i < \infty$ are some constants. This proves the validity of the statement.

Let us formulate the conditions under which all σ_ν and, consequently, the real parts of all roots of Eq. (2.5) are, by virtue of the foregoing nonpositive. For this we shall examine function $R(\sigma, \nu)$ in (2.8). It can be readily shown that $\partial R / \partial\nu < 0$. Hence for $-1 \leqslant \nu \leqslant 1$ curves $R(\sigma, \nu)$ lie in the upper half-plane within the band bounded by curves $R(\sigma, 1)$ and $R(\sigma, -1)$ of which the first lies for all σ under the second and vanishes for $\sigma = \sigma_0 = (\ln|\chi'\varphi'|)/\tau$. Along both branches of that curve R increases monotonically with increasing σ When $|\sigma| \to \infty$, $R = R(\sigma, \nu) \to \infty$ for all curves. All points along the intersection of the indicated band with the straight line $R = 1$ correspond by virtue of the foregoing to roots of the characteristic equation. All relevant configurations appear in Fig. 1, where the position of the OR-axis in this case of stable flow is shown by the solid line, while that of the unstable one is shown by dash line . This implies that the condition of stability (the nonpositiveness of σ_ν) consists of the simultaneous satisfaction of the two inequalities

$$\tau'\sigma_0 \equiv \ln|\chi'\varphi'| < 0, \qquad R(0, 1) \equiv (1 - |\chi'\varphi'|) / |\chi\varphi| \geqslant 0$$

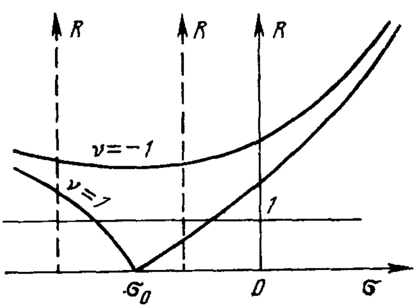

Fig. 1

which are equivalent to (2.6).

Summarizing the foregoing analysis we can state that the simplified characteristic equation (2.5) yields for the stability region boundary a structure that is irregular with respect to τ'/τ or, which is the same, to U_-. For irrational τ'/τ the stability region coincides with rhombus (2.6). When $\tau'/\tau = p/q$ is rational (p/q is an irreducible fraction), then, depending on the values of p and q, some sections of the stability region boundary do not generally coincide with the sides of that rhombus and lie outside it. Without going further into the consideration of this situation we point out that in all such cases it is reasonable to take the smallest stability region (with allowance for conditions of any particular real problem and for the possibility of an "uncontrollable" variation of its determining parameters). In the case considered here this yields rhombus (2. 6).

The determination of the Nyquist curves together with results that are valid for $|\lambda| \gg 1$ does not yield complete information about the disposition of region D (0), the stability region of the characteristic input equation (2. 2). Such information is provided by the stability conditions derived in [3] for the case in which only one of the waves reaching the cross section at $x = 1$ is reflected from it. For $\chi' = 0$ these conditions consist of the requirement that the inequalities

$$|\varphi_0| < 1, \quad p < 1 - \varphi_0, \quad p < -q < \sqrt{\alpha^2 (1 - \varphi_0^2) + p^2} \qquad (2.12)$$
$$(p = -\tau\beta Y, \quad q = \tau Y\ (\beta\varphi_0 - \mu\psi_0))$$

be satisfied simultaneously. Here for $p \neq 0$, α is taken to be the root of the equation $\sin\alpha\ /\ (\cos\alpha - \varphi_0) = \alpha\ /\ p$, while for $p = 0$ it is assumed that $\alpha = \arccos\varphi_0$. In both cases α is within the interval $0 < \alpha < \pi$.

Conditions (2. 12) and their analog for $\chi = 0$ make it possible to determine the stability region of Eq. (2. 2) in the $U_-\chi$- and $U_-\chi'$-planes, as was done in [3] (where by oversight the dash-dot lines in Figs. 2 and 3 are incorrectly represented, which does not, however, affect the remaining results of that paper). The determination of the stability region boundaries in the $\chi\chi'$-plane reduces to finding the smallest region which is not intersected by the Nyquist curves (2. 3) and (2. 4), lies inside rhombus (2. 6), and contains segments of axes χ and χ', which according to (2. 12) or analogous inequalities correspond to stable modes in the case of $\chi = 0$.

3. Computations of the flow of gas with $\varkappa = 1.4$ in a widening channel were carried out as an example of the determination of stability regions by the described method. Some of the obtained results are shown in Figs. 2 and 3.

Boundaries of the stability region in the $\chi\chi'$-plane are plotted in Fig. 2 for $Y = 0.1$ and various U_-, i. e. the intersections of the surface which bounds the stability region in the space $U_-\chi\chi'$, by the planes $U_- =$ const. The velocity U_- upstream of the shock was varied between 1. 4 and 2. 4 in steps of 0. 2. Values of these velocities are indicated at some of the curves. For χ and χ' which correspond to points within the related polygon the flow is stable, while outside it it is unstable. Fig. 2 and similar figures obtained for other values of determining parameters make it possible to solve the question of stability or instability of flow for any specific values of reflection coefficients. Values of χ and χ' which are obtained at constant Mach number and constant rate of flow at $x = 1$ were investigated. The values of χ and χ' determined by the first of these conditions ensure the stability of flow in the channel in all cases, while the specification of constant rate of flow yields χ and χ' close to the boundary of the stability region within and outside the latter, depending on the value of U_-.

The effect of the channel shape (in our approximation, the parameter Y) on the configuration of the stability region is shown in Fig. 3 by solid lines ($U_- = 1.8$ and the numerals denote values of Y). It will be seen that variation of Y affects only one of the five boundaries of the polynomial. The remaining sections of the boundary in the con-

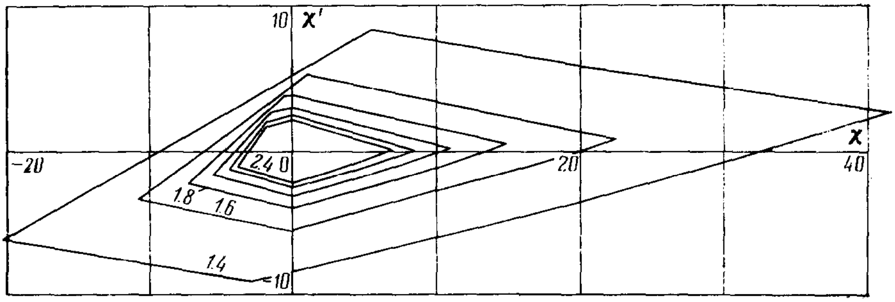

Fig. 2

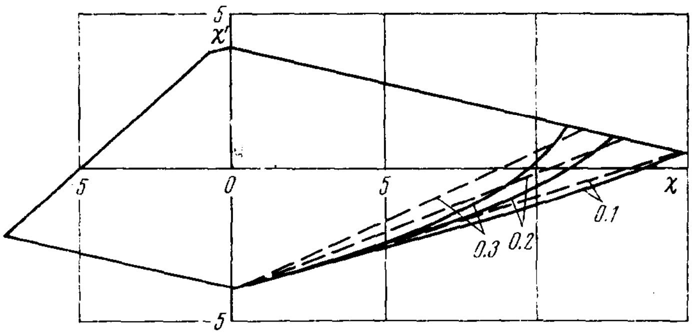

Fig. 3

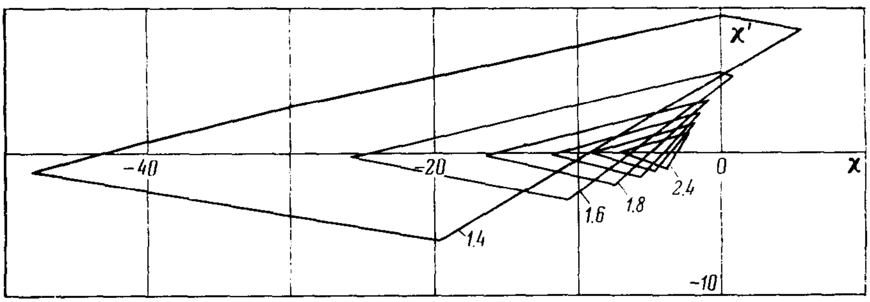

Fig. 4

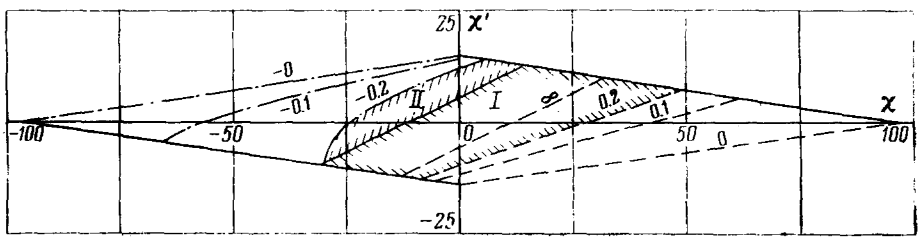

Fig. 5

sidered example coincide either with the singular straight line (2.4) or with boundaries of rhombus (2.6), and are independent of Y. Note that this pattern changes with decreasing U_-. Thus for $U_- = 1.1$ the variation of Y results in the deformation of all boundaries, except the upper left-hand one which consists of the singular straight line (2.4).

4. According to the definitions of τ and τ' and (1.3) we have $\tau'/\tau = (1 + M_+)/2M_+$, which for $U_- = 1$ is equal unity; it increases monotonically with increasing U and for $\varkappa = 1.4$ and $U_- = U_{\max}$ reaches the value of 1.82. We introduce $\tau^\circ = (\tau + \tau')/2$. For $\varkappa = 1.4$ the difference between the ratios τ / τ° and τ' / τ° does not exceed in absolute value 0.3. Hence it is reasonable to expect that the substitution in the complete equation (2.2) of τ° for τ and τ' will not result in a substantial deformation of those sections of the stability boundary which in the "D-separation" method are determined by moderate values of ω. For $\omega \gg 1$, when such simplification is inadmissible (since such substitution is equivalent to the neglect of the product $(\tau - \tau')\omega$ which is not unity), the results based on the analysis of the simplified characteristic equation (2.5) are valid.

In accordance with the above we set in (2.2) $\tau = \tau^\circ$ and $\tau' = \tau^\circ$, and obtain the characteristic equation with a single lag

$$(\lambda + \beta Y)(1 - \varphi^\circ e^{-\tau^\circ\lambda}) + \mu Y \psi^\circ e^{-\tau^\circ\lambda} = 0 \qquad (4.1)$$

$$(\varphi^\circ = \chi\varphi + \chi'\varphi', \quad \psi^\circ = \chi\psi + \chi'\psi')$$

This equation which differs from that considered in [3] only in notation admits the use of conditions of the kind (2.12) without constructing Nyquist curves. The stability region is defined by the intersection of the stability region of Eq. (4.1) and rhombus (2.6).

Without going into details of computations, we present the comparison of results of such approximate method with those of the method described in Sects. 1 — 3, and also some examples of computations.

The effectiveness of the approximate method is fairly clearly observed in Fig. 3, where the dash lines indicate the stability region boundaries obtained by the approximate theory. The dash line sections are shown only where they differ from the solid line ones. A good agreement between the two methods is evident. The difference tends to diminish with decreasing U_-, while it increases (although insignificantly) with increasing U_-. This comparison provides a fairly good justification for using the approximate method. Some of the results obtained by this method are shown in Figs. 4 and 5.

The boundaries of the stability region for a contracting channel $(Y = -0.1)$ are shown in Fig. 4 constructed on the same principle as Fig. 2. Stability regions for widening $(Y > 0)$ and for narrowing $(Y < 0)$ channels appear in Fig. 5 for $U_- = 1.3$. Two sides of rhombus (2.6) and the segment of the singular straight line that represent sections of the stability region boundaries and are independent of Y, are shown by continuous lines. The boundary sections whose shape depends on Y are shown by dash and dash-dot lines. The first of these relate to widening and the second to contracting channels. Related values of Y are indicated by numerals. In Fig. 5 the stability regions for a widening channel with $Y = 0.2$ (region I) and for a contracting channel with $Y = -0.2$ (region II) are shaded on the inside of boundaries. To show more clearly the effect of Y on the shape of the stability region boundaries, the related boundaries for $Y = \infty$.

are plotted in that figure. In this case all of these are straight line segments with the boundary drawn by the dash line parallel to the singular straight line.

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ON THE BOUNDARY LAYER ON A PARTLY MOBILE SURFACE

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Transition of an incompressible boundary layer from the stationary section of a streamlined surface to its mobile section is considered under conditions of stabilized flow. Owing to the motion of a part of the surface a discontinuity of boundary conditions occurs at the surface. It is assumed that the presence of singularity in the boundary conditions does not affect the first approximation boundary layer upstream of the discontinuity line. The problem thus stated was first considered by Mager [1], who obtained an approximate solution for the simplest case of flow past a plate with the unperturbed stream in the form of a Blasius flow and the aft section of the plate moving perpendicularly to the basic stream.

The aim of this paper is the derivation of a solution of equations of the boundary layer in the neighborhood of the discontinuity line on the mobile section in the general case of three-dimensional flows. The solution upstream of that line is assumed known. The method used here may be considered as a generalization of the method of continuation [2] to the case of the three-dimensional boundary layer. A similar scheme of solution derivation for two-dimensional problems of a compressible boundary layer was proposed in [3].